THE

AMERICAN

MATHEMATICAL MONTHLY.

Entered at the Post-Office at Kidder, Missouri, as Second-Class Mail Matter.

VOL. II. MAY, 1895. No. 5.

BIOGRAPHY.

LOBACHEVSKY.

BY DR. GEORGE BRUCE HALSTED.

NICOLAI IVANOVICH LOBACHEVSKY was born November 3, 1793, which, according to the old style calender still used in Russia, is there written October 22, 1793. His father was an architect. This is explicitly stated in the edition of Lobachevsky's works published by the Imperial University of Kasan, so that C. S. Peirce is wrong when in his review of my translation of Vasiliev's Address on Lobachevsky, in the Nation of April 4th, 1895, he criticises Ch. Rumelin for this statement. Rumelin is right. Lobachevsky was born in the town of Makariev about 40 miles above Nizhni-Novgorod on the Volga.

His father died in 1797, and his mother soon after removed to Kasan, where she succeeded in getting her two sons admitted free to the Gymnasium. Lobachevsky entered in November 1802. On February 14th, 1807, after standing his examination, he was admitted as a free student to the University of Kasan, the statues creating which had only been signed November 5th, 1804. Lobachevsky's wildness, disobedience and contempt for orders drew down upon him the severe disapproval of the University authorities; once he was threatened with exclusion from the University, and it was only due to the protection of Bartels, the professor of mathematics, that he was permitted to finish his course.

Toward Bartels Lobachevsky retained to the end of his life the keenest feeling of regard and gratitude.

He studied practical astronomy with Littrow, under whose direction he made observations on the comet of 1811. July 10th, 1811 he received the

NICOLAI IVANOVICH LOBACHEVSKY.

master's degree, and then began teaching as assistant to Bartels. In 1814 he was made adjunct professor of mathematics, and in 1816 full professor. On May 3rd, 1827, when only 33 years old, he was made Rector, and occupied during 19 years the first place in the University of Kasan. In 1846 he was appointed assistant Curator of the district of Kasan, and went to live in a village which belonged to him, Belovoljskaya Slobodka, sixty versts from Kasan up the Volga, where the forest of nut trees planted by him still remains. Toward the end of his life he became blind, but continued his scientific activity and his complete conviction of the profound importance of his non-Euclidean geometry. His last work "Pangeometrie" was produced after his blindness In 1856 he died.

The researches of Lobachevsky on the systematic interpretation of geometry began before 1823, for in that year he presented to Magnetsky, then Rector, with the idea of having it printed at the charge of the crown, a manual of geometry, written in the "classic" form.

It is agreat pity that this most interesting manuscript is lost. From the detailed judgement of Fus on it we cannot gather that Lobachevsky expressed in this manual any original views on the theory of parallels. This point will be of still greater interest in the future, for I am at present engaged in translating from the Hungarian or Magyar language important documents recently obtained in regard to the two Bolyais, and with them a letter, hitherto unknown and not even yet published, from John Bolyai, the scientific twin of Lobachevsky, in which he announces to his father in 1823 the discovery of the non-Euclidean geometry.

Though Lobachevsky's "Geometric Researches on the Theory of Parallels," published in 1840, of which my English translaton is now in its fourth edition and has been beautifully reproduced in Japan, remains even today the simplest introduction to the subject which has ever appeared; yet in it Lobachevsky has not reached that final breadth of view given first in John Bolyas's "Science Absolute of Space," but also attained in Lobachevsky's last work *Pangeometrie*, which name he explicitly uses as expressive of this final view.

There is one point, incomprehensible to his contemporaries, which we can appreciate now as showing the marvellous precision and keenness of Lobachevsky's logic and mathematical perception.

As early as 1834 he mad the distinction in regard to functions, which nearly half a century later Weierstrass and P. du Bois-Reymond forced upon the attention of the mathematical world, namely the distinction between continuity and differentiability. Lobachevsky said, "The function is 'postepennost' [what we now call *continuous*] when the increment in it is diminished to zero together with the increment of the variable x. The function is 'neprerivnost' [what we now call differentiable] when the ratio of these two increments, as they diminish, goes over insensibly into a new function, that will be, consequently, a differential co-efficient." C. S. Peirce says of this, "Who in Russia in 1834 could possibly see anysense in the contention of Lobachevsky that it

was one thing for a curved line to be continuous, and quite another for it to have definite tangents? The mathematicians of Western Europe did not become aware of the distinction until nearly 1880, when Weierstrass suggested that a line might be wavy, and these waves carry smaller waves, and those still smaller waves, and so on *ad infinitum*. Down to this day there is but one text-book on the differential Calculus (that of Camille Jordan, in its second edition) which introduces the distinction. All of Lobachevsky's writings are marked by the same high-strung logic."

The solar corona, one of the most remarkable phenomena in nature, was not enough noticed to receive a name until 1851. But it was carefully observed by Lobachevsky at the eclipse of July 8, 1842, and by him minutely described These things are mentioned to show that Lobachevsky was a modern scientists of the very soundest sort, whose only misfortune was to be half-a-century ahead of the world. But as soon as the world reached him it did ample justice to his influence and his memory. In my Bibliography of non-Euclidean Geometry in the American Journal of Mathematics 1878, reproduced by Vastchenko-Zacharchenko in his Introduction to the Elements of Euclid in 1880, and again reprinted in 1886 at the end of the second volume of the collected geometric works of Lobachevsky (Edition of the Imperial University of Kasan), I gave more than a hundred and seventy five works, and in less than ten years, 1887, the number had grown to over three hundred, and now is so enormously great that the task of a new edition of my Bibliography overwhelms me.

The most distinguished men of the generation just passed, Grassmann, Riemann, Helmholtz, Clifford, Cayley; the ablest of living mathematicians, Lie, Klein, Sylvester, Sir R. Ball, Poircare have won some of their choicest honors in the domain of the non-Euclidean geometry. Its day of probation is safely passed, and one might better square the circle and invent perpetual motion than make the slightest objection to the non-Euclidean geometry.

And now in bringing to a close this meagre notice of a wonderful genius, let me say that no attempt has here been made to give an intimate picture of the man himself, because that has already been done to perfection in the magnificent Address of Lobachevsky's worthy successor at Kasan, Professor A. Vasiliev. Of this L. E. Dickson, of the University of Chicago, himself agenius, speaks as follows: "From every one devoted to mathematics or philosophy, or indeed to the highest advance of human thought in any form, this address will call forth the deepest admiration for Lobachevsky, now recognized as one of the greatest intellectual revolutionizers the world has ever had. It will arouse a deeper enthusiasm for scientific achievement and widen the horizon of every reader."

THE "CATTLE PROBLEM." BY ARCHIMEDIES 251 B. C.

By A. H. BELL, Hillsboro, Illinois.

Compute, O stranger! the number of of cattle of Helios, which once grazed on the plains of Sicily, divided according to their color, to wit:

1st White Bulls$=\frac{\text{Black Bulls}}{2}+\frac{\text{Black Bulls}}{3}+$ Yellow Bulls.

2nd Black Bulls$=\frac{1}{4}$ and $\frac{1}{5}$ of the Dappled Bulls+the Yellow.
3rd Dappled Bulls$=\frac{1}{6}$ and $\frac{1}{7}$ of the White Bulls+the Yellow Bulls.
4th The White cows$=\frac{1}{3}$ and $\frac{1}{4}$ of the Black Herd,Bulls and Cows=Herd.
5th The Black cows$=\frac{1}{4}$ and $\frac{1}{5}$ of the Dappled Herd.
6th The Dappled cows$=\frac{1}{5}$ and $\frac{1}{6}$ of the Yellow Herd.
7th The Yellow cows$=\frac{1}{6}$ and $\frac{1}{7}$ of the White Herd.

He who can answer the above is no novice in numbers. Nevertheless he is not yet skilled in wise calculations! but come consider also all the following numerical relations between the Oxen of the Sun.

8th If the White Bulls were combined in one total, with the Black Bulls they would be in a figure equal in depth and breadth and the far stretching plains of Thrinacia would be covered by the figure (square) formed by them.

9th Should the Yellow and Dappled Bulls be collected in one place, they would stand, if they ranged themselves one after another completing the form of an equilateral triangle. If thou discover the solution of this at the same time; if thou grasp it with thy brain; and give correctly all the numbers: O Stranger! go and exult as conqueror; be assured that thou art by all means proved to have abundant of knowledge in this science.—This is translated by T. L. Heath, author of Diophantos, Cambridge, England, 1889.

The first known answer to the Celebrated Cattle Problem by Archimedes 251 B. C. was computed by the Hillsboro, Illinois, Mathematical Club, 1889 to 1893. Edmund Fish, Geo. H. Richards,and A. H. Bell.

The numbers satisfying all of the 9 conditions as given are the very smallest that will meet the requirements and critical tests that are also given. Mathematicians have heretofore obtained the 8th condition which requires the White and Black Bulls to equal a square number, and is 79 450 446 596 004 $=\square$ number; the 9th condition that the Dappled and Yellow Bulls should equal a triangular number is not fullfilled by the corresponding number, 51, 285 802 909 803, which is designated by B. We seek a square multiplier which call x^2 let $Bx^2=\frac{n\,(n+1)}{2}=$ the expression for a triangular number which gives $8Bx^2+1=(2n+1)^2=y^2$ and we at once get $\sqrt{8B}=\frac{\sqrt{(y^2-1)}}{x}$. The square root of $8B$ by continued fractions will give x, and then we have

$x^2 =$ 34 555 906 354 559 370 506 303 802 963 617 + 68 829 periods of 3's + 252 058 980 100.

White Bulls 1 596 510 804 671 144 531 435 526 194 370 + 68 834 periods of 3's + 385 150 341 800.

Black Bulls 1 148 971 387 728 289 999 712 359 821 824 + 68 834 periods of 3's + 899 825 178 600

Dappled Bulls 1 133 192 754 438 638 077 119 555 879 202 + 68 834 periods of 3's + 921175 894 000

Yellow Bulls 639 034 648 230 902 865 008 559 676 183 + 68 834 periods of 3's + 635 296 026 300

White Cows 1 109 829 892 373 319 039 723 960 215 824 + 68 834 periods of 3's + 914 059 564 000

Black Cows 753 594 142 054 542 639 814 429 119 589 + 68 834 periods of 3's + 238 562 645 400

Dappled Cows 541 460 894 571 456 678 023 619 942 106 + 68 834 periods of 3's + 608 963 318 000

Yellow Cows 837 676 882 418 524 438 692 221 984 107 + 68 834 periods of 3's + 116 422 113 700

Total 7 760 271 406 486 818 269 530 232 833 209 + 68 834 periods of 3's + 719 455 081 800

W. and B. Bulls 2 745 482 192 399 434 531 147 886 016 194 + 68 834 periods of 3's + 284 975 520 400

Root of above 1 656 949 665 133 506 668 + 34 414 periods of 3's + 357 460 163 020

D. & Y. = $\triangle$ = 1 772 227 402 669 540 942 128 115 555 385 + 68 834 periods of 3's + 556 471 920 300

Root of $8\triangle + 1$ 3 765 344 502 347 205 884 + 34 414 periods of 3's + 363 134 961 201

These enormous numbers using 206545 figures will make numbers one-half mile long. In the computations to this problem difficulties are encountered at every step, wonderfull discoveries in the properties of vast numbers are disclosed at every turn. A new summation of continued fractions with many novel ways used to obtain the exact figures shown can be had of A. H. Bell, Hillsboro, Illinois.

REMARKS ON SUBSTITUTION GROUPS.

By G. A. MILLER, Ph. D., Professor of Mathematics University of Michigan, Ann Arbor, Michigan.

Introductory. "The mathematics of the twenty-first century may be very different from our own; perhaps the school boy will begin algebra with the theory of substitution groups as he might now but for inherited habits."*

These words imply two important features of substitution groups, viz., their extensive application and their rudimentary character, for a beginners' algebra must in all times possess these features. To these we may perhaps add a third, viz., their usefulness for unification; for we can scarcely expect that much will be added to the beginners' algebra except what tends to show the unity of subjects which otherwise appear distinct and thus to simplify the acquisition of a knowledge of them. The last one of these features is especially prominent in substitution groups and on this account it seems probable that they will increase in favor with the progress of thought towards abstraction and generalization.

The theory of substitution groups does not concern itself with metrical properties nor is it a part of the science of quantity in the ordinary sense. It chiefly investigates correspondencies by means of which we may apply results with respect to one subject to another without going through the work of investigating each seperately. It tends therefore to save thought and thus becomes a new instrument in the hands of the student by means of which he can work not only faster but also with less exertion than he could without employing it. With these general statements in regard to the scope and features of our subject we proceed to some special consideration of its nature and a few important applications. In doing this we shall not presuppose any knowledge of the subject on the part of the reader.

If we take the four numbers $1, -1, \sqrt{-1}, -\sqrt{-1}$ and multiply any one by itself or by another one of them we obtain no new numbers; thus

$$-\sqrt{-1}\times-\sqrt{-1}=-1,\ -\sqrt{-1}\times-1=\sqrt{-1}$$
$$-\sqrt{-1}\times\sqrt{-1}=1,\ -1\times-1=1, \text{ etc.}$$

We therefore say that these four numbers form a *group* with respect to multiplication. If we take 1 and operate upon it by itself we obtain no new number. 1 therefore forms a *subgroup*† of the given group. If we take -1 in a similar way we obtain the subgroup $1, -1$ since $(-1)^2=1, (-1)^3=-1$, etc. By taking either $\sqrt{-1}$ or $-\sqrt{-1}$ in this way we will obtain the entire group, since $(\sqrt{-1})^2=-1, (\sqrt{-1})^3=-\sqrt{-1}, (\sqrt{-1})^4=1, (\sqrt{-1})^5=\sqrt{-1}$, etc. We say therefore that the above group is *generated* by either $\sqrt{-1}$ or $-\sqrt{-1}$ and that $1, -1$ generate the subgroups 1 and $(1, -1)$ respectively.

*From Professor Simon Newcomb's address delivered before the New York Mathematical Society at its annual meeting, December 28, 1893, and published in the *Bulletin* of this society, vol. III., No. 4.

† The terms group and subgroup are only relative. If a subgroup is considered by itself it is called a group and a group may in turn be a subgroup of a larger group.

We may now consider the corresponding *substitution* group

$$1,\ ac.bd,\ abcd,\ adcb.$$

Where $ac.bd$ means that c is substituted for a and a for c, then d for b and b for d; $abcd$ means that b is substituted for a, c for b, d for c and a for d. 1 means that all the letters are replaced by themselves, or, what amounts to the same, that the letters all remain unchanged.

For example, if we operate upon $a+2b+3c+4d$ with $ac.bd$ we obtain $c+2d+3a+4b$ and if we operate upon the same expression with $abcd$ and $adcb$ we obtain $b+2c+3d+4a$, $d+2a+3b+4c$ respectively. From these definitions it follows directly that 1, $ac.bd$ generate the subgroups 1 and $(1, ac.bd)$ respectively, for if we perform the operation indicated by these symbols any number of times and in any order we obtain a result which is equivalent to the operation indicated by one of the symbols in these subgroups. This is the necessary and sufficient condition that a given aggregate of symbols may be a *group* or subgroup. Since $abcd^2$* $=ac.bd$, $abcd^3=adcb$, $abcd^4=1$, $abcd^5=abcd$, etc., it follows that the above group is generated by either of the substitutions $abcd$ or $adcb$.

In the study of substitution groups it is very important to be familiar with quite a number of such groups. We therefore shall give all the groups of the different numbers of letters not exceeding four. It is evident that the only substitution group of two letters is

$$1,\ ab.$$

The following are the only two that involve three letters:

1	abc	ab
	acb	ac
		bc

1	abc
	acb

The following seven are those that involve four letters:

1	abc	$abcd$	ac	$ab.cd$
	acb	$adcb$	ab	$ac.bd$
	abd	$acbd$	ad	$ad.bc$
	adb	$adbc$	bc	
	acd	$abdc$	bd	
	adc	$acdb$	cd	
	bcd			
	bdc			

1	abc	$ab.cd$
	acb	$ac.bd$
	abd	$ad.bc$
	adb	
	acd	
	adc	
	bdc	
	bdc	

1	$ab.cd$	$abcd$	ac
	$ac.bd$	$adcb$	bd
	$ad.bc$		

1	$ab.cd$
	$ac.bd$
	$ad.bc$

1	$abcd$	$ac.bd$
	$adcb$	

1	ac	$ac.bd$
	bd	

1	$ac.bd$

* This means that the operation indicated by $abcd$ is to be performed twice in succession. In general $abcd{=}x$ $abcd.abcd\ldots abcd$. the operation $abcd$ being performed x times in succession. If we operate once upon $a+2b+3c+4d$ with $abcd$ we obtain $b+2c+3d+4a$, on operating again we obtain $c+2d+3a+4b$; the last result would have been obtained by operating with $ac.bd$ upon the first expression.

The largest group of any given number of letters contains all the substitutions which correspond to the permutations of these letters. The number of substitutions in this group is therefore $n!$ and all the other groups of the same number of letters are subgroups of it. The second group, with respect to size contains all the substitutions which correspond to an even number of interchanges of letters, the number of substitutions in this group is $n! \div 2$.

[To be continued.]

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M., (Princeton), Ph. D., (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from the April Number.]

PROPOSITION XVII. *In the hypothesis of acute angle, we can find a perpendicular and an oblique to the same straight which never meet.*

More literally: *If the straight AH stands* (fig. 15.) *at right angles to any one certain straight AB however small: I say that in the hypothesis of acute angle it cannot hold good, that every straight BD, making with AB any acute angle you choose, toward the parts of this AH, will at length meet this AH produced at a finite, or terminated distance.*

PROOF. Join HB. The angle ABH will be acute (Eu. I. 17), because of the right angle at the point A. Now (Eu. I. 23) a certain HD can be drawn toward the parts of the point B, which not cutting the angle AHB makes with this HB an acute angle equal to this acute angle ABH. Then from the point B is let fall to HD the perpendicular BD, which falls toward the parts of the aforesaid acute angle at the point H. Since therefore the side HB is opposite in the triangle HDB to the right angle at D, and likewise in the triangle BAH to the right angle at A; and again in those two triangles equal angles are adjacent to this side HB, which are in the first triangle indeed the angle BHD, and in the latter the angle HBA; also(Eu.I.26) the remaining angle HBD in the former triangle will be equal to the remaining angle BHA in the latter triangle. Wherefore the entire angle DBA will be equal to the entire angle AHD. Now however, neither of the aforesaid equal angles will be obtuse, lest we meet (from the preceding proposition) in one case the now rejected hypothesis of obtuse angle. Nor will

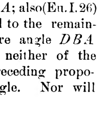


Fig. 15.

either be right, lest we meet (from the same preceding) in one case accordingly the hypothesis of right angle, which (P.V.) will leave no place for the hypothesis of acute angle. Therefore each one of those angles will be acute. This being the case; that the straight BD produced cannot meet in a certain point K this AH produced toward the same parts, is demonstrated thus; because in the triangle KDH, besides the right angle at D, is present the obtuse angle at H, since the angle AHD in the aforesaid hypothesis of acute angle is proved acute. But this is absurd, against Eu. I. 17. Therefore it cannot hold good in this hypothesis, that any BD, making with one straight AB as small as you choose, any acute angle towards the parts of this AH, will at length at a finite, or terminated distance, meet this AH produced. Quad erat demonstrandum.

The same otherwise and more easily. Two perpendiculars AK, BM stand on one certain small at will straight AB (fig. 16). From any point M of this BM let fall the perendicular MH, and join BH. It follows that the angle BHM will be acute. In the hypothesis of acute angle, the angle BMH is also (from the preceding proposition) acute. Therefore the perpendicular BDX, let fall from the point B to this HM, will cut (by Eu. I. 17) this HM in some intermediate point D. Therefore the angle XBA will be acute. But it follows (from the same Eu. I. 17) that those two straights AHK, BDX howsoever produced cannot mutually meet (anyhow at a finite, or terminated distance) on account of the right angles at the points H and D. Therefore in the hypothesis of acute angle it cannot hold good, that any BD, making with one however small straight AB any acute angle toward the parts of this AH, perpendicular to this same AB, will at length meet (at a finite, or terminated distance) this AH produced. Quad erat propositum.

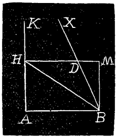


Fig. 16.

SCHOLION I. And this is, what I promised in the Scholia after proposition XIII, that the hypothesis of acute angle (which alone is able now to stand against that general Postutatum of Euclid) will certainly be destroyed by the sole admission of a general meeting of two straights toward those parts, toward which any straight, as small as you choose, meeting them, makes two internal angles less than two right angles; and so indeed, even if either of those angles is to be supposed right.

SCHOLION II. But again in a better place, after proposition XXV, I will show that the hypothesis of acute angle will be equally destroyed, provided that any one acute angle as small as you choose can be designated, under which if any straight meets another, this produced must (at a finite, or terminated distance) finally meet any perpendicular erected upon this meeting straight at whatever finite distance.

PROPOSITION XVIII. *From any triangle ABC, of which* (fig. 17.) *the angle at the point B is inscribed in any semicircle, of which the diameter is AC, is established the hypothesis of right angle, or obtuse angle, or acute angle, according as indeed the angle at the point B was right, or obtuse, or acute.*

PROOF. From the center D join DB. The angles at the base AB will be (Eu. I. 5) equal, and likewise at the base BC, in the triangles ADB, CDB. Wherefore in the triangle ABC the two angles at the base AC will be together equal to the whole angle ABC. Therefore the three angles of the triangle ABC will be together equal to, or greater, or less than two right angles according as the angle at the point B was right, or obtuse, or acute. Therefore from any triangle ABC, of which the angle at the point B is inscribed in any semicircle, whose diameter is AC, is established (P.XV) the hypothesis of right angle, or obtuse angle, or acute angle, according as indeed the angle at the point B is right, or obtuse, or acute.

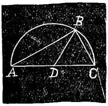


Fig. 17.

Quod erat demonstrandum.

[To be continued.]

AN ATTEMPT TO DEMONSTRATE THE 11th AXIOM OF PLAYFAIR'S EUCLID.

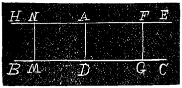
By WARREN HOLDEN, Professor of Mathematics. Girard College, Philadelphia, Pennsylvania.

"*Through a given point one line, and only one, can be drawn parallel to a given line.*"

1st. Two lines perpendicular to a third never intersect, how far soever they be produced. *Halsted's Lobatschewsky's Geometry.* Page 12. Art. 4.

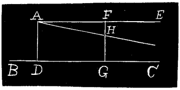
2d. Parallel straight lines are such as are in the same plane, and which, being produced ever so far both ways, do not meet.

3d. Paralles are everywhere equidistant. From A draw AD perpendicular to BC, and through A draw HAE perpendicular to AD. AE and DC being perpendicular to the same line AD are parallel (1st and 2d). From *any* point F let fall the perpendicular FG upon BC. Lay off DM equal to DG, an erect the perpendicular MN. Fold over the part of the figure to the right of AD upon AD as an axis until it falls upon the part to the left. Since A, D, G and M are right angles, and since DM equals DG by construction, AF must fall upon AN, and GF upon MN. The point F is found upon AN and MN, at their intersection N. Therefore GF equals MN. Since F is *any* point, the parallels are everywhere equidistant.



4th. Through a given point one line, and only one, can be drawn parallel

to a given line. Draw AD perpendicular to BC, and through A draw AE perpendicular to AD. Then (1st and 2d) AE and DC are parallel; and the perpendicular FG equals AD, by (3d). Now suppose another line AH parallel to BC. Then HG equals AD or its equal FG. When HG equals FG, AH and AF coincide. Therefore, through a given point one line, and only one, can be drawn parallel to a given line.



The above demonstration may be made without the use of the word parallel. Thus: Through a given point one line, and only one, can be drawn equidistant from a given line.

With the figure drawn as in No. 3, begin the demonstration with the words: From *any* point F let fall the perpendicular &c., to prove the lines equidistant. Then with the same figure as in No. 4, and substituting the word equidistant for parallel, we have the demonstration.

CRADLE-ROCKING BY ELLEPTIC FUNCTIONS.

By F. P. MATZ, M. Sc., Ph. D., New Windsor, Maryland.

After adopting the *gravitation-unit* of force, the equation of motion of the pendulum may be written $(h^2+k^2)(W/g)(d^2\theta/dt^2)=-Wh\sin\theta$....(1). Briefly making $(h+k^2/h)=l$ and $g/l=n^2$, we obtain from (1)

$\frac{1}{2}(d\theta/dt)^2=n^2(\text{vers}\,\alpha-\text{vers}\,\theta)$....(2), in which, according to Sir William Thomson (Lord Kelvin), n is the *angular speed* of the pendulum, Divide *semicircularly* the pendulum-bob, turn downward the convex sides of these divisions centrally joined by a rectilinear axis of inappreciable length, and let the pendulum-rod bisect this rectilinear axis. In the position specified, these divisions constitute the *rockers* of an old-fashioned cradle; and this cradle we regard as placed upon a perfectly rough horizontal plane. Detaching the pendulum rod from the point of suspension, we have to consider *the rocking*, or the rolling oscillations on a horizontal plane, of a material body resting on a semicircular base. Let r=the radius of the equal semicircular rockers. Consider the *line* joining the points of tangency of the rockers with the horizontal plane, as the instantaneous axis of rotation; then, after obvious transformations, (2) becomes $\frac{1}{2}(r^2-2hr\cos\theta+h^2+k^2)(d\theta/dt)^2=gh(\text{vers}\,\alpha-\text{vers}\,\theta)$....(3).

$$\therefore\left(\frac{dt}{d\theta}\right)^2=\frac{[(r-h)^2+k^2]\cos^2\frac{1}{2}\theta+[(r+h)^2+k^2]\sin^2\frac{1}{2}\theta}{4gh(\sin^2\frac{1}{2}\alpha-\sin^2\frac{1}{2}\theta)},$$

$$=\frac{[(r-h)^2+k^2]\cos^2\frac{1}{2}\theta+[(r+h)^2+k^2]\sin^2\frac{1}{2}\theta}{4gh[\sin^2\frac{1}{2}\alpha-(1-\sin^2\frac{1}{2}\alpha)\tan^2\frac{1}{2}\theta]},$$

$$=\frac{[(r-h)^2+k^2]+[(r+h)^2+k^2]\tan^2\frac{1}{2}\theta}{4gh\cos^2\frac{1}{2}\alpha(\tan^2\frac{1}{2}\alpha-\tan^2\frac{1}{2}\theta)}\ldots.(4).$$

In order to transform (4), put $\tan\frac{1}{2}\theta=\tan\frac{1}{2}\alpha\cos\phi$; then differentiating,

$$\text{etc.},\left(\frac{d\theta}{d\phi}\right)^2=4\left(\frac{\tan\frac{1}{2}\alpha\sin\phi}{1+\tan^2\frac{1}{2}\alpha\cos^2\phi}\right)^2,=4\left(\frac{\sin\frac{1}{2}\alpha\cos\frac{1}{2}\alpha\sin\phi}{1-\sin^2\frac{1}{2}\alpha\sin^2\phi}\right)^2\ldots.(a).$$

$$\therefore\left(\frac{dt}{d\phi}\right)^2=\frac{[(r-h)^2+k^2]\cos^2\frac{1}{2}\alpha+(1-\sin^2\phi)[(r+h)^2+k^2]\sin^2\frac{1}{2}\alpha}{gh(1-\sin^2\frac{1}{2}\alpha\sin^2\phi)^2}\ldots.(5).$$

Put $$\varkappa^2=\frac{[(r+h)^2+k^2]\sin^2\frac{1}{2}\alpha}{[(r+h)^2+k^2]\sin^2\frac{1}{2}\alpha+[(r-h)^2+k^2]\cos^2\frac{1}{2}\alpha}\ldots.(b),$$

and $$\varkappa'^2=\frac{[(r-h)^2+k^2]\cos^2\frac{1}{2}\alpha}{[(r+h)^2+k^2]\sin^2\frac{1}{2}\alpha+[(r-h)^2+k^2]\cos^2\frac{1}{2}\alpha}\ldots.(c);$$

then will $\varkappa^2+\varkappa'^2=1$. Make $n=\surd(g/r)$, and represent the denominators of (*b*) and (*c*) by M; then (5) may be written

$$ndt=\sqrt{\left(\frac{M}{hr}\right)}\left[\frac{(1-\varkappa^2\sin^2\phi)d\phi}{(1-\sin^2\frac{1}{2}\alpha\sin^2\phi)\surd(1-\varkappa^2\sin^2\phi)}\right]\ldots.(6).$$

According to the Jacobian system of notation as modified by Gudermann (*Theorie der Modular Functionen*), we have $\phi=\operatorname{am}U$, and $\sin\frac{1}{2}\alpha=\varkappa\operatorname{sn}A$.

Since $\operatorname{dn}^2A+\varkappa^2\operatorname{sn}^2A=1$, we obtain $\operatorname{dn}^2A=1-\varkappa^2\operatorname{sn}^2A=1-\varkappa^2$ $(\sin^2\frac{1}{2}\alpha/\varkappa^2)=\cos^2\frac{1}{2}\alpha$; also,

$$\operatorname{sn}A=\sqrt{\left(\frac{[(r+h)^2+k^2]\sin^2\frac{1}{2}\alpha+[(r-h)^2+k^2]\cos^2\frac{1}{2}\alpha}{[(r+h)^2+k^2]}\right)}\ldots.(d),$$

and $$\operatorname{cn}A=\surd(1-\operatorname{sn}^2A),=\frac{2\cos\frac{1}{2}\alpha\surd(hr)}{\surd[(r+h)^2+k^2]}\ldots.(e).$$

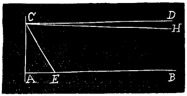
$$\therefore\ ndt=2\left(\frac{\operatorname{sn}A\operatorname{dn}A}{\operatorname{cn}A}\right)\left[1-\frac{\varkappa^2\operatorname{cn}^2A\operatorname{sn}^2U}{1-\varkappa^2\operatorname{sn}^2A\operatorname{sn}^3U}\right]dU\ldots.(7),$$

and $nt=2[(\operatorname{sn}A\operatorname{dn}A/\operatorname{cn}A)\,U-\Pi(U,A)]\ldots.(8)$, while $\tan\frac{1}{2}\theta=\tan\frac{1}{2}\alpha\operatorname{cn}U$.

II. LOGICAL DEDUCTIONS FROM THE HYPOTHESIS THAT THE ANGLE SUM IS LESS THAN TWO RIGHT ANGLES.

By JOHN N. LYLE, Ph. D., Professor of Natural Science in Westminster College, Fulton, Missouri.

Erect the perpendiculars AB and CD to the straight line AC at the points A and C, On AB lay off $AE=AC$ and draw a straight line from C to E.



By construction the triangle ACE is isosceles. Hence, the angle $AEC=ACE$.

By hypothesis the angle sum of every triangle and, hence, of ACE is supposed to be less than two right angles. In accordance with this assumption let the angle sum of the triangle ACE be equal to two right angles—a. Construct $DCH=a$. Then $CAE+ACE+AEC=CAE+ACH$.

Subtract $CAE+ACE$ from both members. Then $AEC=ECH$.

But AEC and ECH are alternate angles. Hence, CH is parallel to AB in the Euclidian sense, that is, it will not meet AB however far both lines may be produced. Euclid. Book I. Proposition XXVII.

Therefore, when the angle sum is assumed in any instance to be equal to $CAE+ACH$, the line CH can not consistently with that hypothesis meet AB.

If, however, the angle sum is assumed to be greater than $CAE+ACH$, it is consistent with this hypothesis to suppose that the line CH may meet AB. For if we make this supposition a triangle will be formed whose angle sum is greater than $CAE+ACH$.

Further, it is inconsistent with the hypothesis to suppose that the line CH can not meet AB. For to deny that CH can meet AB is to deny that a triangle whose angle sum is greater than $CAE+ACH$ can be formed, which is to deny the hypothesis.

But the deduction that CH may meet AB contradicts the conclusion that CH can not meet AB. Therefore, the hypothesis that the angle sum may be greater than $CAE+ACH$ is inconsistent with the hypothesis that the angle sum in any instance is equal to $CAE+ACH$.

That is, the hypothesis that the angle sum is a variable less than two right angles approaching two right angles as a limit is contradictory and hence absurd.

If the hypothesis that the angle sum is less than two right angles is false, sound science requires that the logical deductions from the hypothesis should likewise be false

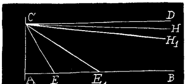
One deduction is that the lines AB and CH making angles with AC whose sum is less than two right angles do not meet. This contradicts Euclid's axiom 12.

Another deduction is that the alternate angles AEC and ECD are not equal although the perpendiculars AB and CD to AC are parallel to each other and can not meet.

Another deduction is that through the same point two straight lines may be drawn parallel to the same straight line. This contradicts the statement known as Playfair's axiom.

Still another of these deductions is that if one side of a triangle be produced the exterior angle is greater than the sum of the two interior and opposite angles.

Lay off $EE_1 = CE$ and draw a straight line from C to E_1.



By hypothesis $ECE_1 + EE_1C + CEE_1$ <2 right angles. But $AEC + CEE_1 = 2$ right angles. Hence $ECE_1 + EE_1C + CEE_1 < AEC + CEE_1$, and $ECE_1 + EE_1C < AEC$.

Add $EAC + ACE$ to both members of this inequality. Then $E_1AC + ACE_1 + AE_1C$ $< EAC + ACE + AEC$. That is, the angle sum of ACE_1 is less than that of ACE.

Let the angle sum of $ACE_1 = 2$ right angles$-b$. But the angle sum of $ACE = 2$ right angles$-a$.

Hence, $b > a$.

Construct $DCH_1 = b$.

Then $CAE_1 + ACE_1 + AE_1C = CAE_1 + ACH_1$, in which $ACH_1 < ACH$.

Subtract $CAE_1 + ACE_1$ from both members. Then $AE_1C = E_1CH_1$.

But AE_1C and E_1CH_1 are alternate angles. Hence, CH_1 is parallel to AB in the Euclidian sense, that is, it will not meet AB however far both lines may be produced. Euclid. Book I. Proposition XXVII.

Therefore, when the angle sum is assumed in any instance to be equal to $CAE_1 + ACH_1$, the line CH_1 can not consistently with that hypothesis meet AB.

The angle sum of ACE is assumed to be $CAE + ACH$, that is, greater than $CAE_1 + ACH_1$.

If greater, the line CH_1 may consistently with the hypothesis meet AB.

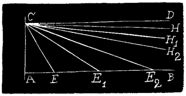
But the deduction that CH_1 may meet AB contradicts the conclusion already reached that CH_1 can not meet AB.

Therefore, the hypothesis that the angle sum may be greater than $CAE_1 + ACH_1$ is inconsistent with the hypothesis that the angle sum in any instance is equal to $CAE_1 + ACH_1$. That is, the hypothesis that the angle sum is a variable less than two right angles approaching two right angles as a limit in value is a contradiction and is therefore false.

Let us proceed with our investigation. Construct the successive isosceles triangles CE_1E_2, CE_2E_3,&c.

Let us consider the series of triangles AE_2C, AE_2C,&c. From the hypothesis that the angle sum is less than two right angles, the conclusion is reached by a process used above in this article that the angle sum of each triangle is less than that of the preceeding triangles in the series.

Draw the lines CH_2, CH_3,&c., making $CAB+ACH_2$ equal to the angle sum of the triangle AE_2C, and $CAB+ACH_3$ equal to the angle sum of the triangle AE_3C,&c. It then follows that $DCH_1 < DCH_2$ and $DCH_2 < DCH_3$,&c.



These results contradict Euclid. They seem also to be inconsistent with each other, for they apparently teach that the lines CH, CH_1, CH_2, CH_3,&c., both do and do not meet AB. Furthermore the inconsistency is interminable inasmuch as the series DCH, DCH_1, DCH_2, DCH_3,&c., is non-terminating.

Lobatschewsky in ennunciating his doctrine of "Imaginary Geomotry" expressly calls his triangle "rectilineal." The logical, geometrical and metaphysical difficulties that follow the denial of the Euclidian axiom 12 and the Euclidian angle sum are so great, however, that non Euclidian writers are now maintaining that Lobatschewsky's triangle can not be drawn in a Euclidian plane and that it is not in fact rectilineal. Since this homeless, outcast triangle is unable to find a "local habitation" in the space of the Alexandrian geometer, the non-Euclidians have excogitated a space especially to contain it called by them "pseudo spherical." Helmholtz in his Lecture "On the origin and significance of geometrical axioms" refers to a "pseudo spherical surface" as "saddle-shaped." He says that the Italian Mathematician E. Beltrami investigated its properties and gave it the name pseudo spherical. Later on in his Lecture he dexterously passes from the phrase —"pseudo spherical surface" to pseudo spherical space." This performance is plainly pseudological. Surface manifestly is not identical with space. Surfaces may be located in space but should not be confounded with space. Beltrami contributes to modern geometrical literature the expression "pseudospherical surface." Helmholtz treats it as identical with "pseudo spherical space" by pseudo logical reasoning and pseudo philosophical speculation.

ARITHMETIC.

Conducted by B.F.FINKEL, Kidder, Missouri. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

44. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor. Maryland.

A, B, and C together bought a ship. A paid for the a/bth, $=\frac{2}{9}$th, part of the ship. B paid for the m/nth, $=\frac{3}{5}$th, part of the ship. C paid \$$M$, $=$\$2000. How many dollars did A, and B, pay?

I. Solution by E. R. ROBBINS, Master of Mathematics in Lawrenceville Schools, Lawrenceville, N. J., COOPER D. SCHMITT, Professor of Mathematics, University of Tennessee, Knoxville, Tennesee, and the PROPOSER.

Since C paid for the $[1-(a/b+m/n)]$th *part* of the ship, the *amount* A paid must be

$$A=\left(\frac{an}{b(n-m)-an}\right)\text{of } \$M, =\left(\frac{1}{b/a(1-m/n)-1}\right) \text{ of } \$M, =\$2500;$$

and, consequently, the amount B paid must be

$$B=\left(\frac{bm}{b(n-m)-an}\right)\text{of } \$M, =\left(\frac{1}{n/m(1-a/b)-1}\right) \text{ of } \$M, =\$6750.$$

NOTE.—The generalized expression for the cost of the ship becomes

$$S=\left(\frac{bn}{b(n-m)-an}\right)\text{of } \$M, =\left(\frac{1}{(1-m/n)-a/b}\right) \text{ of } \$M, =\$11250.$$

II. Solution by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia, and W. I. TAYLOR, Instructor in Mathematics, Berea, Ohio.

$\frac{2}{9}+\frac{3}{5}=\frac{10+27}{45}=\frac{37}{45}$; $\frac{45}{45}-\frac{37}{45}=\frac{8}{45}$, C's share. $\frac{8}{45}=\$2000$, $\frac{1}{45}=\frac{1}{8}$ of $\$200=\$250$. $\frac{10}{45}=10\times\$250=\$2500$, what A, pays. $\frac{27}{45}=27\times\$250=\$6750$, what B, pays.

III. Solution by J. F. W. SCHEFFER, A. M., Hagerstown, Maryland.

C pays for the $1-\left(\frac{a}{b}+\frac{m}{n}\right)$ part of the ship; hence, the price of the ship is $M\div\left[1-\left(\frac{a}{b}+\frac{m}{n}\right)\right]$. A's share $=\frac{a}{b}\cdot\frac{M}{1-(a/b+m/n)}=\$2500$.

$$B\text{'s}=\frac{m}{n}\cdot\frac{M}{1-(a/b+m/n)}=\$6750.$$

Also solved by *P. S. Berg*.

45. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

In running a mile, A can give B $a=20$ yards; B can give C $b=88$ yards. How may yards can A give C?

I. Solution by P. S BERG, Apple Creek, Ohio.

1 mi. $-$ $\frac{1}{88}$ mi. $=$ $\frac{87}{88}$ mi., distance B runs while A runs a mile.

1 mi. $-$ $\frac{1}{20}$ mi. $=$ $\frac{19}{20}$ mi., distance C runs while B runs a mile. $\frac{87}{88}\times\frac{19}{20}=\frac{1653}{1760}$mi. distance C runs while A runs a mile.

Hence 1 mi $-$ $\frac{1653}{1760}$ mi $=$ $\frac{107}{1760}$ mi $=$ 107 yards, the distance A can give C.

II. Solution by E. L. SHERWOOD, Houston, Mississippi.

A runs a mile while B runs 1740 yards. B runs a mile while C runs 1672 yards or C runs $\frac{1672}{1760}$ of B's distance. So A runs 1760 yards; B, 1740 yards, and C, $\frac{1672}{1760}$ of 1740 yards or 1653 yards.

Whence A can allow C 1760 yards$-$1653 yards or 107 yards.

This problem was also solved by *Cooper D. Schmitt*, *W. I. Taylor*, *G. B. M. Zerr*, *J. F. W. Scheffer* *E. R. Robbins*, and the *Proposer*.

PROBLEMS.

50. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

If A walk to the city and ride back, he will require $m=5\frac{1}{4}$ hours; but if he walk both ways, he will require $n=7$ hours. How many hours will he require to ride both ways?

51 Proposed by F. P. MATZ. M. Sc., Ph. D., Professor of Mathematics and Astronomy, in New Windsor College, New Windsor, Maryland.

A banker, in discounting a note due in $m=4$ months at $r=3\%$ per annum, charges $C=\$12\frac{1}{4}$ more than the true discount. What is the face of the note discounted?

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

41. Proposed by A. H. BELL, Hillsboro, Illinois.

In a right-angled triangle there are given, the bisectors of the acute angles. Required the triangle.

I. Solution by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland, and D. G. DORRANCE, Jr., Camden, N. Y.

Represent the half-angles by α and $(45-\alpha)$; then easily is deduced

$$\tan 2\alpha = n\cos(45-\alpha)/m\cos\alpha \;\ldots\ldots(1).$$

$$\therefore\quad \tan^3\alpha+\tan^2\alpha+[(2m\sqrt{2}-n)/n]\tan\alpha-n=0\;\ldots\ldots(2);$$

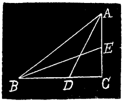
$$\text{that is, } (\tan\alpha-Q_1)(\tan\alpha-Q_2)(\tan\alpha-Q_3)=0\;\ldots\ldots(3).$$

Hence three sets of values of the sides of the required right triangle are possible. *Numeralizing* m and n in (2), we deduce Q_1, Q_2, and Q_3 from (3); then α is known. Consequently the three sides, $b=m\cos\alpha$, $p=n\cos(45-\alpha)$, and $h=m\cos\alpha\sec2\alpha$, are known.

II. Solution by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Let $BC=x$, $AC=nx$, $AB=mx$, $AD=a$, $BE=b$.

Then $m^2-n^2=1\ldots\ldots(1)$.



$$ma+na=2mnx\cos\tfrac{1}{2}A=\frac{2mn^2x^2}{a}\;\ldots\ldots(2).$$

$$b+mb=2mx\cos\tfrac{1}{2}B=\frac{2mx^2}{b}\;\ldots\ldots(3).$$

$$(2)\div(3)\text{ gives }\frac{(m+n)a^2}{(m+1)b^2}=n^2\;\ldots\ldots(4).$$

Eliminating n between (1) and (4),

$$m^6+2m^5-\left(1+\frac{2a^2}{b^2}\right)m^4-\left(4+\frac{2a^2}{b^2}\right)m^3-\left(1-\frac{2a^2}{b^2}\right)m^2+2\left(1+\frac{a^2}{b^2}\right)$$

$$m+\frac{a^4}{b^4}+1=0.$$

Let $\dfrac{a}{b}=u$. Then $m^6+2m^5-(1+2u^2)m^4-(4+2u^2)m^3-(1-2u^2)$

$m^2+2(1+u^2)m+u^4+1=0.$

To give a more complete solution of this equation might be interesting but well-nigh impossible unless we use numerical results. Such a solution, however, is as unsatisfactory as the problem itself.

Let $u^2=\frac{128}{81}$. Then $3m=5$ or $m=\frac{5}{3}$, $n=\frac{4}{3}$, $x=3$.

$\therefore$ $mx=5$, $nx=4$, $x=3$. Let $u^2=4$. Then $m=1.332$, $n=.8799$, $x=.936-$. $\therefore$ $mx=1.246$, $nx=.8235$, $x=.936$, when $a=2$, $b=1$.

Let $a=40$, $b=50$, $u^2=\frac{16}{25}$, Then $m=1.2532$, $n=.7553$, $x=47.4072+$.

$\therefore$ $mx=59.4107+$, $nx=35.8067+$, $x=47.4072+$.

Let $a=b=c$, then $u^2=1$, $m=\sqrt{2}$, $n=1$, $x=\frac{c}{2}\sqrt{(2+\sqrt{2})}$.

$$\therefore\quad mx=\frac{c}{\sqrt{2}}\sqrt{(2+\sqrt{2})},\ nx=x=\frac{c}{2}\sqrt{(2+\sqrt{2})}.$$

III. Solution by B F. BURLESON, Oneida Castle, New York

Let ABC be the triangle, right angled at C. Put $AD=a=40$, and $BE=b=50$, the lines bisecting the acute angles A and B. Put $x=AB$, $y=AC$, and $z=BC$. Put $\phi+\theta=$the$\angle CAD$ and $\phi-\theta=$the $\angle CBE$. We have, by Trigonometry,

$$x=b\cos(\phi-\theta),\ldots.(1),$$
$$y=a\cos(\phi+\theta),\ldots.(2),$$
$$y=z\tan(2\phi-2\theta)\ldots.(3),$$

Eliminating from (1), (2), and (3), we obtain by development

$(b+b\tan\phi\tan\theta)\left(\dfrac{1-\tan^2\theta-2\tan\theta}{1-\tan^2\theta+2\tan\theta}\right)=0\ldots.(4)$. This is true because $\phi=22\frac{1}{2}^\circ$.

Clearing (4) of fractions, resolving factors, and substituting for $\tan\phi=22\frac{1}{2}^\circ$ its equal $\sqrt{2}-1$, observing that $\cot\phi=\sqrt{2}+1$, we get $(b+a)\tan^2\theta+\{(b-a)(\sqrt{2}+1)+[2(b-a)]\}\tan^2\theta+\{2(b+a)(\sqrt{2}+1)-(b+a)\}\tan\theta=(b-a)(\sqrt{2}-1)\ldots.(5)$. Dividing (5) by $b+a$ and substituting the numerical values of a and b, we get

$\tan^3\theta-490468\tan^2\theta+3.828427125\tan\theta=.268245951375$. Hence, by Horner's Method of Detached Coefficients, $\tan\theta=.0693633$, and the auxiliary angle $\theta=3^\circ 58' 4\frac{7}{8}''$. By substituting in (1) and (2), we determine that $y=35.807338$ and $z=47.407325$. $\therefore$ $x=\sqrt{(y^2+z^2)}=59.410604$.

This problem was also solved by *A. H. Bell*, *J. F. W. Scheffer*, and *H. C. Wilkes*.

42. Proposed by ALEXANDER MACFARLANE, A. M., D.Sc, LL. D., Cornell University, Ithaca, New York.

There are p electors and q candidates for r seats. Each elector has r votes, and he may distribute them as he pleases among the candidates. Find in how many different ways the voting may result, that is, the number of possible states of the poll.

Solution by G. B. M. ZERR, Staunton, Virginia, and F. P. MATZ, New Windsor, Maryland.

The number of different ways of voting for r seats out of q candidates, when each elector casts r votes for r different persons, is

$$n=\frac{q(q-1)(q-2)(q-3)\ldots.(q-r+1)}{1.2.3.4\ldots.r}.$$

If $p > n$, then, since there can be but n different ways of voting, n will be the number of different ways the voting may result.

If $p < n$, then since p persons can prepare only p states of the poll, p will be the number of different ways the voting may result.

Also solved by *H. C. Whitaker*.

GEOMETRY.

Conducted by B.F.FINKEL, Kidder, Missouri. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

40. Proposed by J. C. CORBIN, Pine Bluff, Arkansas.

If R, r, r_1, r_2, and r_3 be, respectively, the radii of the circumscribed, inscribed, and escribed circles of a $\triangle$, prove $r_1+r_2+r_3-r=4R$.

Solution by M. A. GRUBER. War Department, Washington, D. C.

From any $\triangle$ whose sides are a, b, and c, we obtain $R=\dfrac{abc}{4\triangle}$,

$r=\dfrac{\triangle}{s}$, $r_1=\dfrac{\triangle}{s-a}$, $r_2=\dfrac{\triangle}{s-b}$, and $r_3=\dfrac{\triangle}{s-c}$.

$$\therefore \quad r_1+r_2+r_3-r=\frac{\triangle}{s-a}+\frac{\triangle}{s-b}+\frac{\triangle}{s-c}-\frac{\triangle}{s}=\frac{2s^3-as^2-bs^2-cs^2+abc}{\triangle}$$

$$=\frac{s^2[2s-(a+b+c)]+abc}{\triangle}=\frac{abc}{\triangle}. \quad \text{But } \frac{abc}{\triangle}=4R. \quad \therefore \quad r_1+r_2+r_3-r=4R.$$

We might appropriately add a few other combinations of these radii.

$$(1)\ \frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{1}{r};\ (2)\ rr_1r_2r_3=\triangle^2;\ (3)\ Rrr_1r_2r_3=\frac{abc\triangle}{4}.$$

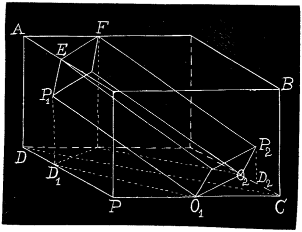
Solutions of this problem were received from *G. I. Hopkins*, *E. W. Morrell*, *P. S. Berg*, *G. B. M. Zerr*, *F. P. Matz*, *Cooper D. Schmitt*, *P. H. Philbrick*, *J. F. W. Scheffer*, *John B. Faught*, and the *Proposer*. H. C. Whitaker did not solve the problem but referred to Chauvenet's Geometry and Hallowell's Geometrical Analysis, p. 225.

41. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Find the length (x) of a rectangular parallellopiped b=5ft., and h=3ft., which can be *diagonally inscribed* in a similar parallelopiped L=83ft., B=64ft, and H=50ft.

Solution by B. F. BURLESON, Oneida, Castle, New York, and the PROPOSER.

Let $x=AF$, $y=AE$, $z=P_2D_2$, and $l=O_1P_1=$the required length of the inscribed rectangular parallelopiped; then, obuiously, $x^2+y^2=b^2$ ....(1), $(L-x)^2+(B-y)^2+(H-z)^2=l^2$ ....(2), $x(L-x)=y(B-y)$ .....(3), and $h_1\ [(L-x)^2+(B-y)^2]=lz$ ....(4). From (3) and (1), $4y^4-4By^3+(B^2-4b^2+L^2)y^2+2Bb^2y=(L^2-b^2)b^2$ ....(5); and this with coefficients numerically expressed, becomes $4y^4-256y^3+10885y^2+3200y=171600$ .....(6).



Therefore, by *Horner's Method of Approximation*, we have from (6), $y=4$; whence $x=3$. Briefly putting the now known $(L-x)^2+(B-y)^2,=m^2=10000$, we have from (2) and (4), respectively, $m^2+(H-z)^2=l^2$ ....(7), and $lz=hm$ ....(8). Therefore, $l^4-(H^2+m^2)l^2+2Hhml=h^2m^2$ ....(9); that is, $l^4-12500l^2+30000l=90000$ ....(10).

Whence $l=110.617130324415$ feet.

Cor.—Make $H=0$, and $h=0$; then the problem becomes: *Find the length of a rectangle of given width inscribed diagonally in a given rectangle.*

After performing obvious operations, we obtain

$l^4-(B^2+2b^2+L^2)l^2+4BbLl=(B^2-b^2+L^2)b^2$ ....(11); or with the coefficients numerically expressed, we have the equation,

$$l^4-11035l^2+106240l=274000 \text{ ....(12)}.$$

Therefore $l=100$ feet, which is the length of the diagonally-inscribed rectangle required.

A. H. Bell gets 107.5 feet as a result

42. **Proposed by G I. HOPKINS, Instructor in Mathematics and Physics in High School, Manchester, New Hampshire,**

If the bisectors of two angles of a triangle are equal the triangle is isosceles.

Solution by the PROPOSER.

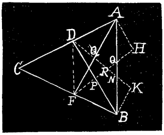
Let AF and BD bisect the angles of the triangle ABC, and let $AF=BD$.

Draw DF. Make $\angle PDO=\angle PDF$, and $\angle QFN=\angle QFD$.

Draw AH perpendicular to AF and BK perpendicular to BD.

Draw FH through O and DK through N.

$\triangle DFB = \angle DOB$, having two angles and included side of one etc. $\therefore$ $BF=BO$. $\therefore$ BP is perpendicular to OF, for a line which bisects the vertical angle of an isosceles triangle is perpendicular to the base. Similarly $AD=AN$, and AQ is perpendicular to ND. $\triangle$'s DPR and FQR are right-angled at P and Q. $\therefore$ $\angle RDP = \angle QFR$. $\therefore$ $\triangle$'s AHF and KBD are equal, since they are right-angled at B and A, and have a leg and adjacent acute angle of one equal respectively to a leg and adjacent acute angle of the other.



$\therefore$ $AH=KB$. BK is parallel to HF, and AH is parallel to KD, being perpendicular to the same line. $\therefore$ $\angle KBN = \angle HOA$, and $\angle KNB = \angle HAO$, being exterior interior angles. $\therefore$ $\angle H = \angle K$.

$\therefore$ $\triangle$'s KNB and HAO are equal, having two angles and included side etc. $\therefore$ $AO=NB$. $\therefore$ $AN=OB$. $\therefore$ $AD=BF$. $\therefore$ $\triangle$'s ADF and BDF are equal, having three sides respectively equal.

$\therefore$ $\angle DAF = \angle DBF$, and $\therefore$ $\angle A = \angle B$. $\therefore$ $AC=BC$, being opposite equal angles. Q. E. D.

As this problem is one that has frequently been discussed and is of interest to mathematicians we shall publish, in the June MONTHLY, two or three more of the many excellent solutions we have received. A query from Dr. George Lilley says, "It is said that Mr. I. Todhunter proposed the above problem, and that a direct or *a priori* proof has not been discovered for it. What is the *a priori* proof?—ED.

PROBLEMS.

46. Proposed by GEORGE E. BROCKWAY, Boston, Massachusetts.

If an equilateral triangle is inscribed in a circle, the sum of the squares of the lines joining any point in the circumference to the three vertices of the triangle is constant.

47. Proposed by J. C. GREGG, Superintendent of Schools, Brazil, Indiana.

Given two points A and B and a circle whose center is O: show that the rectangle contained by OB and the perpendicular from B on the polar of A is equal to the rectangle contained by OB and the perpendicular from A on the polar of B.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

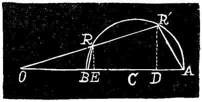
SOLUTIONS OF PROBLEMS.

31. Proposed by E. B. ESCOTT, Ann Arbor, Michigan.

Through a point O on the produced diameter AB of a semicircle draw a secant ORR', so that the quadrilateral $ABRR'$ inscribed in the semicircle shall be a maximum. Prove that in this case, the projection of RR' on AB is equal in length to the radius of the circle. [*Williamson's Diff. Calculus*, 7th edition, p. 189, Ex. 25.]

I Solution by JOHN B. FAUGHT, A. B., Instructor in Mathematics, Indiana University, Bloomington, Indiana.

Let $OC=p$; then the equation of the circle is $\rho^2-2\rho p\cos\theta+p^2-r^2=0$. The roots of which are $\rho_1=p\cos\theta+\sqrt{(r^2-p^2\sin^2\theta)}=OR'$ and $\rho_2=p\cos\theta-\sqrt{(r^2-p^2\sin^2\theta}=OR$.



$\triangle OAR'=OA.OR'\sin\theta=(p+r)[p\cos\theta+\sqrt{(r^2-p^2\sin^2\theta}]\sin\theta$.

$\triangle OBR=AB.OR\sin\theta=(p-r)[p\cos\theta-\sqrt{(r^2-p^2\sin^2\theta}]\sin\theta$.

$\therefore$ Qd. $ABRR'=2p[r\sin\theta\cos\theta+\sin\theta\sqrt{(r^2-p^2\sin^2\theta}]$.

$$\therefore\ \frac{d(\text{Qd})}{d\theta}=\frac{r(1-2\sin^2\theta)\sqrt{(r^2-p^2\sin^2\theta)}+\cos\theta(r^2-2p^2\sin^2\theta)}{\sqrt{(r^2-p^2\sin^2\theta)}}=0,$$ for a maximum or minimum.

Reducing we have $4p^2\sin^4\theta-4(p^2+r^2)\sin^2\theta+3r^2=0$.

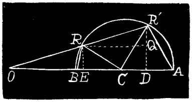
$$\therefore\ \sin\theta=\frac{\sqrt{(r^2+p^2)\pm\sqrt{(r^4-r^2p^2+p^4)}}}{p},$$ which must be a maximum from the nature of the problem.

$$ED=OR'\cos\theta-OR\cos\theta=(\rho_1-\rho_2)\cos\theta=2(\sqrt{r^2-p^2\sin^2\theta)}\cos\theta$$
$$=\sqrt{4(r^2-p^2\sin^2\theta)(1-\sin^2\theta}=\sqrt{4r^2-4(p^2+r^2)\sin^2\theta+4p^2\sin^4\theta}$$
$$=\sqrt{(4r^2-3r^2)}=r.$$

II. Solution by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland, and J. F. W. SCHEFFER, A. M., Hagerstown, Maryland.

Let $CB=r$, $CO=a$, $a/r=c$, $\angle COR=\theta$, $CR'O=\phi$; then $\angle ACR'=(\phi+\theta)$, $\angle OCR=(\phi-\theta)$, $OR'=r\sin(\phi+\theta)/\sin\theta$, $OR=r\sin(\phi-\theta)/\sin\theta$, and

$\sin\phi=c\sin\theta\ldots\ldots(1)$. The area of the quadrilateral is
$\mathbf{A}=\frac{1}{2}[AO+OR'-BO\times OR]\sin\theta, =ar[\cos\theta+\surd(1-c^2\sin^2\theta)]\sin\theta\ldots.(2)$, which is to be a maximum.



Differentiating, etc., we have

$$\frac{d\mathbf{A}}{d\theta}=\frac{1-2\sin^2\theta}{1-2c^2\sin^2\theta}+\frac{\surd(1-\sin^2\theta)}{\surd(1-c^2\sin^2\theta)}=0\ldots.(3).$$

$\therefore\quad 4c^2\cos^4\theta-4(c^2-1)\cos^2\theta-1=0\ldots.(\alpha)$,

or $4c^2\sin^4\theta-4(c^2+1)\sin^2\theta+3=0\ldots.(\beta)$.

From (α), $\cos^2\theta=\frac{1}{2}c^{-2}\{\surd[(c^2-1)^2+c^2]+(c^2-1)\}\ldots.(4)$.

From (1), $\cos^2\phi=\frac{1}{2}\{\surd[(c^2-1)^2+c^2]-(c^2-1)\}\ldots.(5)$.

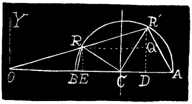
The projection of RR' on AB is $\mathbf{P}=OR'-OR)\cos\theta, =2r\cos\phi\cos\theta\ldots.(6)$.

Substituting the square roots of (4) and (5) in (6), etc., we have $\mathbf{P}=r$.

III. Solution by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Let O be the origin, Ox, Oy the axes, C the centre of the circle, $OC=a$, $BC=AC=r$, $\tan ROB=m$.

Then equation to circle is $(x-a)^2+y^2=r^2$, and equation to line ORR' is $y=mx$.



By substitution we get

$$(x-a)^2+m^2x^2=r^2.$$

$$\therefore\quad x=\frac{a\pm\surd(r^2+r^2m^2-a^2m^2)}{1+m^2},$$

$$y=\frac{ma\pm m\surd(r^2+r^2m^2-a^2m^2)}{1+m^2}.$$

Now area $OR'A-$area $ORB=$max. $\therefore\ \frac{1}{2}OA\times R'D-\frac{1}{2}OB\times RE=$max.

$$\therefore\ \tfrac{1}{2}(a+r)\left\{\frac{ma+m\surd(r^2+r^2m^2-a^2m^2)}{1+m^2}\right\}$$

$$-\tfrac{1}{2}(a-r)\left\{\frac{ma-m\surd(r^2+r^2m^2-a^2m^2)}{1+m^2}\right\}=\text{max}.$$

$$\therefore\ \frac{mar+ma\surd(r^2+r^2m^2-a^2m^2)}{1+m^2}=\text{max}.$$

Reducing the first differential coefficient we get

$$r^2+r^2m^2-2a^2m^2=r(m^2-1)\surd(r^2+r^2m^2-a^2m^2).$$

Squaring and reducing we easily get $m^4r^2+2(2a^2-r^2)m^2=3r^2$.

$\therefore\ m^2=\frac{1}{r^2}[2\surd(a^4-a^2r^2+r^4)-2a^2+r^2]$, as the only admissible value.

The projection of RR' on $AB=ED=OD-OE=\dfrac{2\surd(r^2+r^2m^2-a^2m^2)}{1+m^2}=r$ for the above value of m^2.

When $a=r$, $m^2=1$. $\therefore\ \angle ROB=45°$.

Also solved by *Alfred Hume*, *C. E. White*, and *H. W. Draughon*.

MECHANICS.

Conducted by B.F.FINKEL, Kidder, Missouri. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

17. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Find the law of density of strings collected into a heap at the edge of a table with the end of the string just over the edge, so that equal masses may always pass over in equal units of time.

II. Solution by the PROPOSER.

Let x=the length of string which depends from the edge of the table at the end of the time t from the beginning of motion, k=the density of the string at a unit's distance from the end, and *assume* that the density varies as the nth power of the distance from the end. The mass of the depending length is then $=\int_0^x kx^n dx=\dfrac{k}{n+1}x^{n+1}$, and if a= the mass passing over the edge in a unit of time, and g=the acceleration of gravity, we have for the equation of motion, $\dfrac{d}{dt}\left(\dfrac{k}{n+1}x^{n+1}\dfrac{dx}{dt}\right)=gat\ldots.(1)$,

$$\text{or, }\frac{d}{dt}\left(\frac{k}{n+1}x^{n+1}\frac{dx}{dt}\right)=\frac{k}{n+1}gx^{n+1}\ldots.(2),$$

or, $\dfrac{d}{dt}\left(x^{n+1}\dfrac{dx}{dt}\right)=gx^{n+1}\ldots.(3)$.

Multiplying both members of (3) by $2\left(x^{n+1}\dfrac{dx}{dt}\right)$ and integrating,

$$\left(x^{n+1}\frac{dx}{dt}\right)^2=\frac{2g}{2n+3}x^{2n+3}\ldots.(4),\quad\text{or,}\quad\frac{dx^2}{dt^2}=\frac{2g}{2n+3}x\ldots.(5).$$

Equation (5) gives $\dfrac{d^2x}{dt^2}=\dfrac{g}{2n+3}\ldots.(6)$,

$$\text{whence }\frac{dx}{dt}=\frac{g}{2n+3}t\ldots.(7).$$

From (7), $gat=a(2n+3)\dfrac{dx}{dt}=\dfrac{d}{dt}\left(\dfrac{k}{n+1}x^{n+1}\dfrac{dx}{dt}\right)\ldots.(8)$.

Integrating (8) and dividing by x, $\dfrac{k}{n+1}n^n\dfrac{dx}{dt}=a(2n+3)\ldots.(9)$.

Equation (9) gives $\dfrac{1}{t}\cdot\dfrac{k}{n+1}x^{n+1}=a(n+1)(2n+3)=a\ldots.(10)$,

since the *mass* of string passes over at the uniform rate a.

Equation (10) gives $2n^2+5n=-2$, or $n=-\frac{1}{2}$, or $n=-2$.

19. Proposed by H. C. WHITAKER, B. Sc., M. E., Professor of Mathematics, Manual Training School, Philadelphia, Pennsylvania.

"There was an old woman tossed up in a basket,
Ninety times as high as the moon."
What was her initial velocity, the resistance of the air being neglected?

Solution by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, University, Mississippi, and J. C. NAGLE, M.A., C.E., Professor of Civil Engineering, A and M. College, College Station, Texas.

If g is the acceleration of gravity at the earth, R the surface of the earth's mean radius, and x the distance of the body from the center of the earth at the time t, the equation of motion is $\frac{d^2x}{dt^2}=-\frac{a^2}{x^2}g$.

Integrating, $\left(\frac{dx}{dt}\right)^2=\frac{2a^2}{x}g+c$.

Taking the moon's distance as $60.3R$, $\frac{dx}{dt}=0$ when $x=5427R$.

$\therefore\quad c=-\frac{2R^2}{5427R}g$, and $\left(\frac{dx}{dt}\right)^2=2R^2g\left(\frac{1}{x}-\frac{1}{5427R}\right)$.

When $x=R$, the velocity (the initial velocity required)

$=\sqrt{2Rg\left(\frac{1}{R}-\frac{1}{5427R}\right)}=\sqrt{\frac{5399}{2713.5}Rg}$, which is 6.9+ miles per second.

Also solved by *P. S. Berg*, *E. W. Morrell*, *H. W. Draughon*, *G. B. M. Zerr*, *F. P. Matz*, and the *Proposer*.

NOTE.—Professor Hoover sent a fine solution of problem 18, but it came to late for insertion in April MONTHLY.—EDITOR.

PROBLEMS.

27. Proposed by G. B. M. ZERR, A. M., Principal of High School, Staunton. Virginia.

One thousand balls, each having a mass of 10 grams, and each moving with a velocity of 10 kilometers per second, are confined in a certain space with elastic walls. Into the same space are now introduced one thousand balls each of 100 grams mass, and moving with a velocity each of 10 kilometers per second; collisions take place, and finally, after a number of encounters, the average kinetic energy of each of the two thousand balls is the same. Show that this is 2.75(10)11 in the centimeter-gram-system.

28. Proposed by O. W. ANTHONY, Mexico, Missouri.

A movable finite wire carrying a current os electricity is perpendicular to and on one side of an infinite wire also carrying a current. Invertigate the motion of the movable wire.

DIOPHANTINE ANALYSIS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

21. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy, in New Windsor College, New Windsor, Maryland.

Find (1) nine positive *integral numbers* in arithmetical progression the sum of whose squares is a *square number*; and (2) find nine integral *square numbers* whose sum is a *square number*.

Solution by ARTEMAS MARTIN, LL.D., U S. Coast and Geodetic Survey Office, Washington, D.C.

1. Let $x-4y$, $x-3y$, $x-2y$, $x-y$, x, $x+y$, $x+2y$, $x+3y$, $x+4y$ denote the required numbers in arithmetical progression, and the sum of their squares is $9x^2+60y^2=\square$ ....(1).

Put $y=3z$ and (1) becomes, after dividing by 9,

$$x^2+60z^2=\square=\left(x+\frac{p}{q}z\right)^2....(2);$$

$$\text{whence } \frac{x}{z}=\frac{60q^2-p^2}{2pq}.$$

Let $p=1$, $q=\frac{1}{2}$; then $\frac{x}{z}=\frac{14}{1}$. Take $x=14$, $z=1$; then $y=3$, and the numbers are 2, 5, 8, 11, 14, 17, 20, 23, 26; and we have

$$2^2+5^2+8^2+11^2+14^2+17^2+20^2+23^2+26^2=48^2.$$

Let $p=1$, $q=1$; then $\frac{x}{z}=\frac{59}{2}$. Take $x=59$, $z=2$; then $y=6$, and the numbers are 35, 41, 47, 53, 59, 64, 70, 76, and 80; and we have

$$35^2+41^2+47^2+53^2+59^2+64^2+70^2+76^2+82^2=183^2.$$

By giving suitable values to p and q an infinite number of sets may be found.

2. Take the well-known identity $(x+y)^2=(x-y)^2+4xy$....(3).

If we can transform $4xy$ into a square we shall have two square numbers whose sum is a square. Since x may be any quantity whatever, we may put $x=a+b+c+d+e+f+g+h$, and then we have

$$(a+b+c+d+e+f+g+h+y)^2=(a+b+c+d+e+f+g+h-y)^2$$
$$+4y(a+b+c+d+e+f+g+h)....(4).$$

The last term will be a square if we take $a=i^2$, $b=j^2$, $c=k^2$, $d=l^2$, $e=m^2$, $f=n^2$, $g=p^2$, $h=q^2$, $y=r^2$, and we have

$$(i^2+j^2+k^2+l^2+m^2+n^2+p^2+q^2+r^2)^2$$
$$=(i^2+j^2+k^2+l^2+m^2+n^2+p^2+q^2-r^2)^2.$$

$+(2ri)^2+(2rj)^2+(2rk)^2+(2rl)^2+(2rm)^2+(2rn)^2+(2rp)^2+(2rq)^2 \ldots\ldots (5)$. Take $i=1), j=2, k=3, l=4, m=5, n=6, p=7, q=8, r=9$; then, after dividing the numbers by 3, we have $6^2+12^2+18^2+24^2+30^2+36^2+41^2+42^2+48^2=95^2$.

An infinite number of sets of nine square numbers whose sum is a square may be found from (5).

27. Proposed by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Decompose into the sum of two squares the product $5\times13\times61$.

I. Solution by DAVID E. SMITH, Ph. D., Professor of Mathematics, State Normal School, Ypsilanti, Michigan.

Taking the usual formula

$(m^2+n^2)(x^2+y^2)=(mx\pm ny)^2+(my\mp nx)^2$, and noticing that $5.13.61=65.61$, the problem reduces to decomposing 65 and 61 into two squares, which give

$$65=64+1=49+16,$$

$$61=36+25.$$

$$\therefore\quad 65.61=53^2+34^2=43^2+46^2=62^2+11^2=22^2+59^2.$$

The products 1.3965, 5.793, and 13.305 give no different results.

II. Solution by the PROPOSER.

$(m^2+n^2)(p^2+q^2)=(mp\pm nq)^2+(mq\mp np)^2, =A^2+B^2.$

$(m^2+n^2)(p^2+q^2)(r^2+s^2)=(A^2+B^2)(r^2+s^2), =(Ar\pm Bs)^2+(As\mp Br)^2.$

Let $m=2$, $n=1$, $p=3$, $q=2$, $r=5$, $s=6$.

Then $A^2+B^2=(6\pm2)^2+(4\mp3)^2, =8^2+1^2, =4^2+7^2.$

$(2^2+1^2)(3^2+2^2)(5^2+6^2)=5\times13\times61$

$=(40\pm6)^2+(48\pm5)^2, =(20\pm42)^2+(24\mp35)^2.$

$=46^2+43^2, =34^2+53^2, =62^2+11^2, =22^2+59^2.$

III. Solution by COOPER D. SCHMITT, Professor of Mathematics, University of Tennessee, Knoxville, Tennessee.

I present a general solution of this problem, true for any number found according to the same law.

$5=1^2+2^2$, $13=(1+2)^2+(1\times2)^2, =3^2+2^2$. $61=(3+2)^2+(3\times2)^2=5^2+6^2$.

Then in general we have $(x^2+y^2)[(x+y)^2+(xy)^2][(x+y+xy)^2+(xy)^2(x+y)^2]$ and this is to be shown to consist of two squares.

Using determinants we have as follows:

$$\begin{vmatrix} x & -y \\ y & x \end{vmatrix} \times \begin{vmatrix} x+y & -xy \\ xy & x+y \end{vmatrix} \times \begin{vmatrix} x+y+xy & -x^2y-xy^2 \\ x^2y+xy^2 & x+y+xy \end{vmatrix}$$

$$= \begin{vmatrix} x^2+xy+xy^2, & x^2y-xy-y^2 \\ xy+y^2-x^2y, & xy^2+x^2+xy \end{vmatrix} \times \begin{vmatrix} x+y+xy & -x^2y-xy^2 \\ x^2y+xy^2 & x+y+xy \end{vmatrix}$$

$$\left| = \begin{matrix}(x^2+xy+xy^2)(x+y+xy)+(x^2y-xy-y^2)(-x^2y-xy^2), \\ (xy+y^2-x^2y)(x+y+xy)+(xy^2+x^2+xy)(-x^2y-xy^2),\end{matrix}\right.$$

$$\left.\begin{matrix}(x^2+xy+xy^2)(x^2y+xy^2)+(x^2y-xy-y^2)(x+y+xy) \\ (xy+y^2-x^2y)(x^2y+xy^2)+(xy^2+x^2+xy)(x+y+xy)\end{matrix}\right|$$

$$=(x^3+2x^2y+2x^2y^2+xy^2+xy^3+x^3y+3x^2y^3+xy^4+x^3y^2-x^4y^2-x^3y^3)^2$$
$$+(x^4y+3x^3y^2+x^3y^3+x^2y^3+x^2y^4+x^3y-x^2y-2xy^2-y^3-xy^3)^2.$$

If $x=1$, $y=2$, as in given problem, I get $5\times13\times61=59^2+22^2$.

If $x=2$, $y=3$, I find $13\times61\times2021=398^2+807^2$ and so on indefinetly.

AVERAGE AND PROBABILITY.

Conducted by B.F.FINKEL, Kidder, Missouri. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

14. Proposed by CHARLES E. MYERS, Canton, Ohio.

$\frac{1}{3}$ of all the melons in a patch are not ripe, and $\frac{1}{4}$ of all the melons in the same patch are rotten, the remainder being good. If a man enters the patch on a dark night and takes a melon at random, what is the probability that he will get a good one?

I. Comment by JOHN DOLMAN, Jr., Counsellor-at-Law, Philadelphia, Pennsylvania.

The published solution of Probability problem No. 14 is erroneous. The simplest correct solution is as follows:

If the melon selected is *not unripe* and *not rotten*, it will be good. The chance that it is unripe is $\frac{1}{3}$, therefore the chance that it is not unripe is $\frac{2}{3}$; by similar reasoning, the chance that it is not rotten is $\frac{3}{4}$. Therefore the chance that it is not unripe and not rotten, is $\frac{2}{3}$ of $\frac{3}{4}=\frac{1}{2}$, which is the chance required.

It is not very difficult to point out the error in the published solution. While it is true there cannot be more than $8n$ nor less than $5n$ good melons, it does not follow that $\frac{1}{2}(8n+5n)$ is the average or most probable number, unless it be predicated that all values between $5n$ and $8n$ are equally likely, which is not the case.

Dropping the n's, suppose there are 8 ripe and 4 unripe melons, the three rotten ones may be selected from these 12 in 220 different ways each of equal probability. Now a very simple application of the principles of choice will show that of these ways

4 would leave 8 good melons
48 would leave 7 good melons
112 would leave 6 good melons
56 would leave 5 good melons

giving therefore the numbers 5, 6, 7, and 8 the relative values thus found, and averaging in the ordinary way, we will find that 6 is the average number of good melons, and therefore $\frac{6}{12}$ or $\frac{1}{2}$ is the probability of selecting a good one.

15. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy, in New Windsor College, New Windsor, Maryland.

Todhunter proposes: "From a point in the circumference of a circular field a projectile is thrown at random with a given velocity, which is such that the diameter of the field is equal to the greatest range of the projectile; prove the chance of its falling within the field, is $C=2^{-1}-2\pi^{-1}(\sqrt{2}-1),=.236+$." Is this result perfectly correct as to fact?

Comment on the Solution of Problem 15, by JOHN DOLMAN Jr., Philadelphia, Pennsylvania.

I should not presume to criticise the work of so able and celebrated a mathematician as Professor Matz, did I not consider his solutions of Probability No. 15 vicious in their effects upon the minds of students, striking as they do at the root, not only of the doctrine of mean value and probability, but of the integral calculus itself. "Since the projectiles are *thrown* at random they should *fall* at random," is on a par with, "as an arc varies uniformly the sine varies uniformly," or "because acceleration is constant velocity is constant." The third solution is not clear, but how far any given range the favorable chances can be represented by an area, when the projectiles must fall on the arc of a circle, is difficult to understand. Also in the fifth solution it is stated," for any range PD', the projectiles falling on the circular arc DMD' are within the field" and then the angle PAD' is adopted as uniformly varying, without giving any reason for it. However interesting this may be as mathematical legerdemain, its effects are vicious when it is published without proper explanation for in my humble opinion it is more important that your readers learn to reason correctly than that they be taught to integrate ingeniously.

NOTE.—The solution of problem 14 was published without comment for the reason that we considered the solution to be correct, and we confess that we do not yet see the force of Mr. Dolman's argument, though we have not had time to give it much thought.

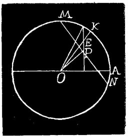
As to the solutions of problem 15, we hold that the first solution is the only correct solution as that one and that one alone involves the strict literal statement of the problem. It is evident that the number of ways the projectile can be thrown is equal to the surface of a hemisphere whose radius is R. If now we find the surface of that part of this hemisphere any point at which if a projectile be thrown the projectile will fall upon the circular field, diameter R, and then divide this surface by the surface of the hemisphere, the result will be the probability required. This method of solution would have to be accepted by the most critical mind. Professor Matz's first solution involves this principle.—EDITOR.

19. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in New Windsor College, New Windsor, Maryland.

Find the average area of the circle which is the locus of the middle points of all chords passing through a point taken at random in the surface of a given circle.

I. Solution by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Let P be the random point. Through P draw HG perpendicular to OA. Let $OA=a$, $GP=x$, $\angle HOA=\theta$, area circle required$=\frac{\pi}{4}OP^2=\frac{\pi}{4}(x^2+a^2\cos^2\theta)$. An element of the circle at P, is $a\sin\theta d\theta dx$. The limits of θ are 0 and 2π; of x, 0 and $a\sin\theta=x'$. $\triangle$=average area.



$$\therefore\ \triangle=\frac{\int_0^{2\pi}\int_0^{x'}\frac{\pi}{4}(x^2+a^2\cos^2\theta)a\sin\theta d\theta dx}{\int_0^{2\pi}\int_0^{x'}a\sin\theta d\theta dx}$$

$$=\frac{1}{4a}\int_0^{2\pi}\int_0^{x'}(x^2+a^2\cos^2\theta)\sin\theta d\theta dx=\frac{a^2}{12}\cdot\int_0^{2\pi}(\sin^2\theta+3\cos^2\theta)\sin^2\theta d\theta=\frac{\pi a^2}{8}.$$

II. Solution by the PROPOSER.

Let $OP=x$; then the average area of the circle whose diameter is OP,

$$\text{becomes } A=\int_0^{2\pi}\int_0^{r}(\tfrac{1}{4}\pi x^2)xd\theta dx\div\int_0^{2\pi}\int_0^{r}xd\theta dx=\tfrac{1}{8}\pi r^2.$$

Professor Zerr furnished five different solutions of this problem; Professor Matz seven and Mr. Dolman three.

MISCELLANEOUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

15. Proposed by SAMUER HART WRIGHT, M.D., M.A.,Ph.D., Penn Yan, Yates Co., New York.

Required the illuminated area of the Moon's disc when $\frac{2}{3}$ through its first quarter of 60° of longitude east of the Sun, the Earth and Moon being at their mean distances.

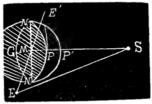
II. Solution by G. B. M. ZERR, A. M., Principal of High School, Staunton. Virginia, and the PROPOSER.

A rigorous solution of this problem would present many difficulties. In the first place, the surface of the Moon is more than half illuminated; in the second, no observer sees half of the Moon's surface at one time, in the third, the mean distance of the Sun is known within a quarter millio■ miles.

In the following solution we will assume half the Moon's surface illuminated, and half of its surface as presented to an observer at one time. Also we will take the Sun's parallax 8.″81 and hence, his mean distance

as 149320000 kilometers, and the Moon's mean distance as 384000 kilometers.

Let S, M, E, be the centres of the Sun, Moon, and Earth respectively. Let $EM=1$, then $SE=388.854$, also $\angle MES=60^\circ$.



From Trigonometry we get

$EM+ES:ES-EM=\tan\frac{1}{2}(EMS+ESM)$
$:\tan\frac{1}{2}(EMS-ESM)$

$389.854:387.854=\tan 60^\circ:\tan\frac{1}{2}(EMS-ESM)$.

$\therefore\ \frac{1}{2}(EMS-ESM)=59^\circ 52' 20''$,

$\therefore\ \angle EMS=119^\circ 52' 20''$.

Now since EM is perpendicular to the plane $N'BNP'$, and MS is perpendicular to the plane $N'GNP$, $\angle P'ME=180^\circ-119^\circ 52' 20''=60^\circ 7' 40''=\theta$.

Let r=radius of the Moon, then MP the semi-conjugate diameter of the ellipse $N'GNP=r\cos\theta$.

Area of bright crescent $N'PNP'N'$=area of semi-ellipse $NMNPN'$ subtracted from semi-circle $N'NPN'$; area semi-circle$=\frac{1}{2}\pi r^2$,

area semi ellipse$=\frac{1}{2}\pi r^2\cos\theta$; and area crescent$=\frac{1}{2}\pi r^2(1-\cos\theta)=.250965\pi r^2$ $=\frac{1}{4}$ of the disc nearly. In the above solution we have regarded the observer as being at the centre of the earth.

16. Yale Senior Prize Problem. Contributed by H. A. NEWTON, LL.D., Professor of Mathematics, Yale College, Mew Haven, Connecticut.

The axis of two right cylinders whose bases are circles of 4 and 6 inches radius respectively, intersect at right angles; compute to four decimal places the lengths of the curves of intersection of the two surfaces.

Solutiou by G. B. M. ZERR, A. M., Principal of High School, Staunton, Virginia.

Let $x^2+y^2=36$, be the equation to the cylinder with base radius 6 inches.

$x^2+z^2=16$, the equation to the cylinder with base radius 4 inches.

$$\text{Then}\ \frac{dy}{dx}=-\frac{x}{\sqrt{(36-x^2)}},\quad \frac{dz}{dx}=-\frac{x}{\sqrt{(16-x^2)}}.$$

$$\therefore\ S=\int\left\{1+\left(\frac{dy}{dx}\right)^2+\left(\frac{dz}{dx}\right)^2\right\}^{\frac{1}{2}}dx=\int\left\{\frac{576-x^4}{(36-x^2)(16-x^2)}\right\}^{\frac{1}{2}}dx.$$

$$\text{The whole length of the curve}=4s=4\int_0^4\left\{\frac{576-x^4}{(36-x^2)(16-x^2)}\right\}^{\frac{1}{2}}dx.$$

Let $x=4\sin\theta$.

$$\text{Then } 4s=16\int_0^{\frac{\pi}{2}}\left\{\frac{9-4\sin^4\theta}{9-4\sin^2\theta}\right\}^{\frac{1}{2}}d\theta=16\int_0^{\frac{\pi}{2}}\left\{\frac{1-e^2\sin^4\theta}{1-e^2\sin^2\theta}\right\}^{\frac{1}{2}}d\theta,\ \text{ where } e=\tfrac{2}{3}.$$

Expanding we get

$$4s=16\int_0^{\frac{\pi}{2}}(1+\tfrac{2}{9}\sin^2\theta-\tfrac{4}{27}\sin^4\theta-\tfrac{16}{729}\sin^6\theta-\tfrac{200}{6561}\sin^8\theta-\tfrac{48}{6561}\sin^{10}\theta$$

$$-\tfrac{1408}{177147}\sin^{12}\theta-\tfrac{3392}{1594323}\sin^{14}\theta-\ldots.)d\theta.$$

$$4s=8\pi(1+\tfrac{1}{9}-\tfrac{1}{18}-\tfrac{5}{729}-\tfrac{875}{104976}-\tfrac{7}{3888}-\tfrac{847}{472392}-\tfrac{53053}{1190427784}-\dots).$$

$4s=26.0104$ exact to two decimal places.

Also solved by *F, P. Matz*, and *J. F. W. Scheffer*.

PROBLEMS.

30. Proposed by R. J. ADCOCK, Larchland, Illinois.

When the sum, of the distances of a point of a plane surface, from all its other points, is a minimum, that point is the centre of gravity of the plane surface.

QUERIES AND INFORMATION.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

"*The Mysterious Formula.*"

Referring to my article on "Logarithms of Negative Numbers" published in the April Number, Vol. I., Mr. C. D. Schmitt, on page 214, deduces the following singular result: $\pi=\dfrac{\log(-1)}{\sqrt{-1}}$.

Another very remarkable result can be deduced from this as follows:

Dividing by 2, we have $\dfrac{1}{2\sqrt{-1}}\log(-1)=\frac{1}{2}\pi$. This may be written $\log\sqrt{-1}^{-\sqrt{-1}}=\frac{1}{2}\pi$. $\therefore\ \sqrt{-1}^{-\sqrt{-1}}=e^{\frac{1}{2}\pi}=4.810477381$. This is what Professor Benjamin Peirce in his linear Associative Algebra, p. 5 (edition published by D. Van Nostrand), calls "the mysterious formula."

Writing i for $\sqrt{-1}$, the formula is $\dfrac{1}{i^i}=\sqrt{e^{\pi}}$.

M. C. Stevens.

Editor Finkel has notified me that he cannot spare the space for further discussion of the possibility of a root of the equation $4+\sqrt{(x-4)}-\sqrt{(x+4)}=0$. I will therefore again refer Mr. Draughon to Mr. Horner's article in the Philosophical Magazine, and also to Wentworth's Algebraic Analysis, page 278 to page 286.

H. C. Whitaker.

The International Mathematical Congress.

Professor A. Vasiliev, President of the Physico-mathematical Society

of Kasan, Russia, has sent me a document prepared by him for the minister of public instruction, with a request that I translate such part of it from the Russian as bears on the founding of an International Mathematical Congress, and make it known in America. This is in substance as follows:

After recapitulating the action of the French Association for the advancement of Science at Caen (August 14,1894)[already translated by me and published on pp. 21-22 of the Bulletin of the American Mathematical Society, October 1894], he gives the resolution offered by me that very same day, August 14, 1894, for their signatures to all of the members of the American Mathematical Society present at the Brooklyn meeting, and signed unanimously, which was as follows: "The undersigned, members of the American Mathematical Society present at its summer meeting, 1894, take this method of expressing their cordial approval of a series of international Congresses of mathematicians to take place from time to time, as suggested by A. Vasiliev and C. A. Laisant." The names of the signers may be found on page 290 of of Vol. I of the MONTHLY. I explained the plan as contemplating a *reunion prepartoire* at Kasan in 1896, and a *congres constituant* in Belgium or Switzerland in 1897, which perhaps might fix the first international congress at Paris in 1900.

Professor Vasiliev then goes on to state the decisive step taken by the *deutsche Mathematiker-Vereinigung* in a reunion at Vienna, September 1894. It was there unanimously resolved to take part in the organizing congress. The action was as follows: "Concerning future international congresses, the Mathematiker-Vereinigung decides in principle to participate, and charges its bureau to take in regard to this subject the measures that appear necessary. In particular, it leaves to each of its members entire freedom, considering alone as essential that the socieiy, on this important occasion, may be assured of having the place due it." Professor Vasiliev expects that the inauguration of the Lobachevsky monument at Kasan will take place in August or September 1896, and counts on having there a large number of eminent mathematicians, and will profit by the occasion to propose definitely the organisation of the international congress, and then official calls will be issued to meet for the purpose of final organization in 1897 at a city of Belgium or Switzerland.

GEORGE BRUCE HALSTED.

Austin, Texas.

I. What explanation do the mathematicians of the *present* day give of the old paradox following, where precisely opposite results are reached by apparently rigorous demonstrations?

If A and B are traveling in the same line, A at the rate of one mile an hour and B, starting one mile behind him and traveling at the rate of two miles per hour, it is evident that in one hour A will have traveled one mile and B, two miles and will have overtaken A, as we know by experience the truth is. But while B is traveling one mile, A will have traveled one half as far, and so on. That is, while B is traveling the space between him and A at the

commencement of an hour, A will have traveled half as far, and hence B can never overtake him.

To the human mind, in the absence of all other knowledge, this demonstration is rigorously correct and conclusive. But we know that the conclusion is absolutely the opposite of the truth. The only explanation formerly given was, that while the human mind cannot comprehend it, the space between A and B finally becomes infinitely small, or, as compared with finite quantities, *nothing*.

J. H. Drummond.

II. It is often stated that it is impossible to trisect an angle with the rule and compass only. Has this impossibility been demonstrated and if so where can the demonstration be found?

W. E. Heal.

EDITORIALS.

We shall consider it a great favor if all subscribers who have not yet paid their subscription for 1895, will kindly remit at once. We need the money to pay the Publishers.

At a recent meeting of the Philosophical Faculty of Yale University, Editor Finkel was assigned a graduate Scholarship for the academical year of 1895-6.

Rudyard Kipling will shortly return to India where he will prepare, for *The Cosmopolitan*, twelve articles to appear in the American and English editions of that magazine. India is one of the most interesting of countries, and Mr. Kipling is able to write of it as no one else. His work will be looked forward to with world-wide expectation.

Leonard E. Dickson, of the Chicago University, was reappointed to a Fellowship and will remain at the University another year.

There are a number of our subscribers in arrears for 1894. We shall consider it a kindness if those who are owing for 1894, will remit the amount of the subscription at once. A mathematical journal of the size and scope of the Monthly can not be published without funds, and were it not for a number of our mathematical friends aiding us financially the Monthly would be obliged to discontinue.

Subscribers who wished their subscription to cease with Vol. I. should have notified us on the receipt of Dec. No. Some after receiving three or four extra numbers, ask us to discontinue, without paying for the extra copies.

Our mathematics does not teach any such principles and should we find among our mathematical collections a book teaching any such doctrine we would consign the same to the flames. One of the very best things that can be claimed for the study of mathematics is that it develops strong tendencies to honesty and justice.

BOOKS AND PERIODICALS.

The Cosmopolitan: An Illustrated Monthly Magazine. Edited by John Brisben Walker and Arthur Sherburne Hardy. Price, $1.50 per year. Single Number 15 cents.

Perhaps the most beautiful series of pictures ever presented of the Rocky mountains will be found in a collection of fourteen original paintings, executed by Thomas Moran for the May Cosmopolitan. To those who have been in the Rockies, this issue of The Cosmopolitan will be a souvenir worthy of preservation. This number contains fifty-two original drawings, by Thomas Moran, Oliver Herford, Dan Beard, H. M. Eaton, F. G. Attwood, F. O. Small, F. Lix, J. H. Dolph, and Rosina Emmett Sherwood, besides six reproductions of famous recent works of art, and forty other interesting illustrations—ninety-eight in all. Though the Cosmopolitan sells for but fifteen cents, probably no magazine in the world will present for May so great a number of illustrations specially designed for its pages by famous illustrators. The fiction in this number is by F. Hopkinson Smith, Gustav Kobbe, W. Clark Russell, Edgar W. Nye, and T. C. Crawford.

The Ascent of Man. By Henry Drummond, LL.D., F.R.S.E., F.G.S. Fourth Edition. 8vo cloth. 346 pp. Price, $2.00. New York: James Potts & Co., Publishers.

This, to my mind, is the crowning glory of Dr. Drummond's writings. Treating as he does one of the profoundest questions in the history of man, he has told the story of man's Ascent as seen in the light of modern science, in a way that can not fail to impress deeply the most unscientific mind. And though the Author, in his preface, modestly says that the theme is Ascent not Decent, and that the book is a Story not an Argument, yet the Story is so full of sound statements that it has all the force of argument. The book is one we trust will fall into the hands of every lover of scientific research. It is truly scientific and bears no marks of skepticism. B.F.F.

The Basis: A Weekly Journal of Citizenship. Edited by Judge Albion W. Tourgee, Mayville, N. Y., and Published by the Citizens' Publishing Co., Buffalo, N. Y. Price, $1.50 per year. 10 cts. per copy.

The first number of this Journal which has for its object the elevation of the people to higher citizenship and the dissemination of sound principles of government, appeared in April, 1895.

As its editor is one of the leading writers, and one of the ablest jurists in this country, and as the objects of the Journal are noble and unselfish, its success is written on every page. It deals with all questions relating to government and society. Its editor is a fearless and uncompromising advocate of common justice and equal rights. The Basis should be in every home in the land. B. F. F.